On the Roots of Lower Binomials

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Roots of a Trinomial

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- Completing the Square

Roots of a Trinomial

How do we find the roots of

$$f(x) = -0.142702 - 0.616906x^{10} + 0.773992x^{39}?$$

Binomials

For small $|x|$,

$$\begin{aligned} f(x) &= -0.142702 - 0.616906x^{10} + 0.773992x^{39} \\ &\approx -0.142702 - 0.616906x^{10} \end{aligned}$$

For large $|x|$,

$$\begin{aligned} f(x) &= -0.142702 - 0.616906x^{10} + 0.773992x^{39} \\ &\approx \qquad\qquad\quad -0.616906x^{10} + 0.773992x^{39} \end{aligned}$$

Binomials

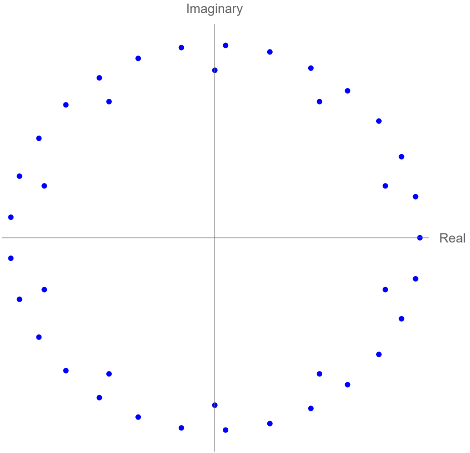
Instead of

$$f(x) = -0.142702 - 0.616906x^{10} + 0.773992x^{39}$$

look at the *binomials*

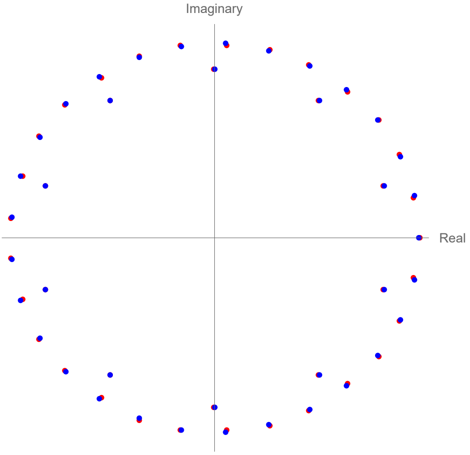
$$f_1(x) = -0.142702 - 0.616906x^{10}$$
and
$$f_2(x) = -0.616906x^{10} + 0.773992x^{39}$$

Roots of the Binomials



Roots of f_1 and f_2

Roots of f and its binomials



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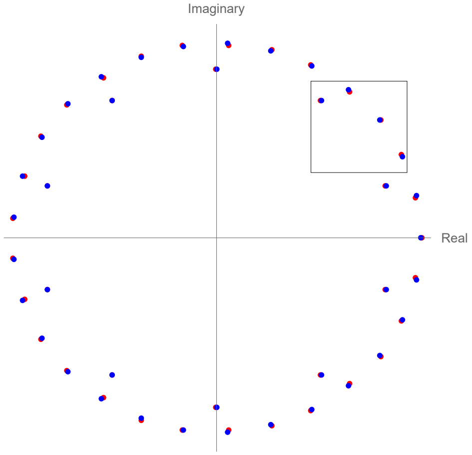

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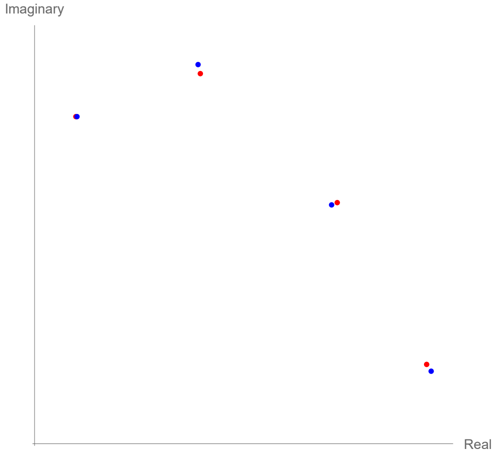

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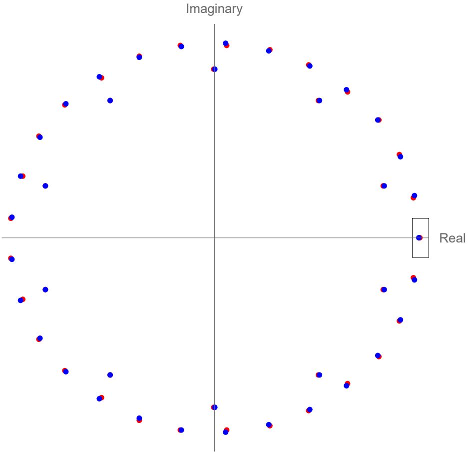

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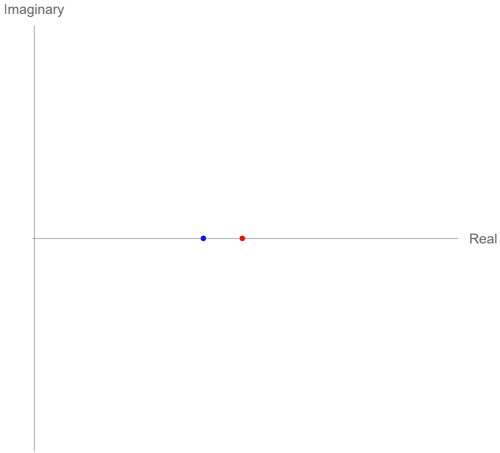

Positive roots of f and binomials f_1 and f_2

Why Binomials?

$$g(x) = c_0 x^{a_0} + c_1 x^{a_1}, \quad \text{for } 0 \leq a_0 < a_1$$

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- We can find their roots explicitly! For $0 \leq k < a_1 - a_0$,

$$x = \sqrt[(a_1 - a_0)]{\left|\frac{c_0}{c_1}\right|} \left(\cos\left(\frac{2\pi k}{a_1 - a_0}\right) + i \sin\left(\frac{2\pi k}{a_1 - a_0}\right)\right).$$

Archimedean Newton Polygon

For $f(x) = c_0 x^{a_0} + c_1 x^{a_1} + c_2 x^{a_2}$ we define

$$\mathrm{ArchNewt}(f) = \mathrm{Conv}\{(a_i, -\log|c_i|) \mid i \in \{0,1,2\}\}.$$

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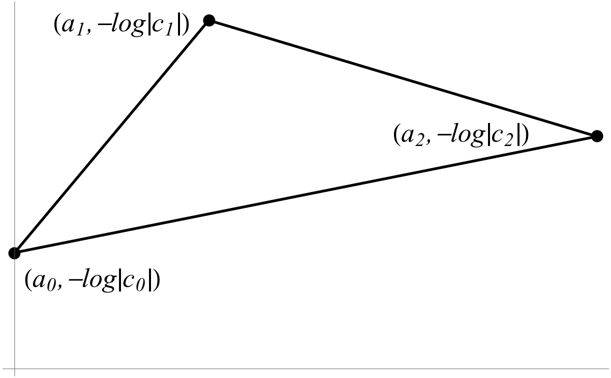
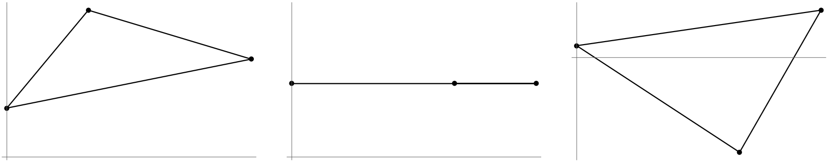


Forms of ArchNewt(f)

Since f is a trinomial, $\mathrm{ArchNewt}(f)$ looks like one of the following triangles.



Lower Binomials

Definition

Consider a univariate trinomial $f \in \mathbb{R}[x]$.

$$f(x) = c_0 + c_1 x^{a_1} + c_2 x^{a_2}.$$

The *lower binomials* of f are the binomial summands that correspond to the lower edges of the $\mathrm{ArchNewt}(f)$.

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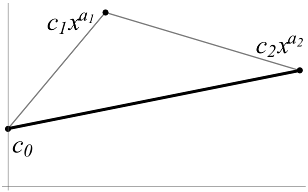

Lower Binomial $f_1(x) = c_0 + c_2 x^{a_2}$

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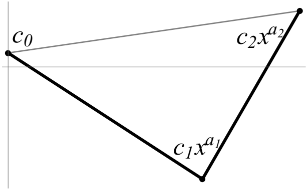

Lower Binomials $f_1(x) = c_0 + c_1 x^{a_1}$ and $f_2(x) = c_1 x^{a_1} + c_2 x^{a_2}$

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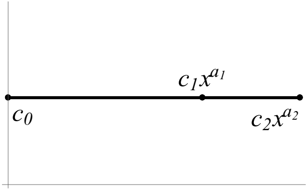


No Lower Binomial

Flatness Condition

Lemma

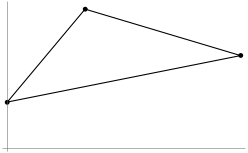
For a given trinomial $f(x) = c_0 + c_1 x^{a_1} + c_2 x^{a_2}$, define the *Flatness Condition*

$$F = \frac{-\log|c_1| + \log|c_0|}{a_1} - \frac{-\log|c_2| + \log|c_0|}{a_2}.$$

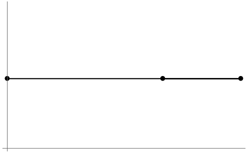
If $F > 0$, the trinomial f has one lower binomial. If $F < 0$, the trinomial f has two lower binomials.

Flatness Condition

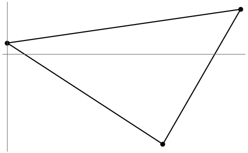
If $F > 0$,



If $F = 0$,



If $F < 0$,



Lower Binomials

Definition

Given a univariate trinomial $f \in \mathbb{R}[x]$,

$$f(x) = c_0 + c_1 x^{a_1} + c_2 x^{a_2}.$$

1. If f has one lower edge ($F > 0$), then $c_0 + c_2 x^{a_2}$ is the lower binomial of f.
2. If f has two lower edges ($F < 0$), then $c_0 + c_1 x^{a_1}$ and $c_1 x^{a_1} + c_2 x^{a_2}$ are the lower binomials of f.

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Lower Binomials

The lower binomials of

$$f(x) = -0.142702 - 0.616906x^{10} + 0.773992x^{39}$$

are

$$f_1(x) = -0.142702 - 0.616906x^{10}$$

and

$$f_2(x) = -0.616906x^{10} + 0.773992x^{39}.$$

Roots of Lower Binomials

Question 1

How can we approximate the roots of a trinomial using the roots of its correspondent lower binomials?

Question 2

How often does a trinomial have as many positive roots as its lower binomials?

Roots of Lower Binomials

Question 1

How can we use the roots of lower binomials to approximate the roots of the trinomial?

Newton's Method

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Define,

$$N(f, z_0) = \begin{cases} z_0 - f(z_0)/f'(z_0) & \text{if } f'(z_0) \neq 0 \\ z_0 & \text{if } f'(z_0) = 0 \end{cases},$$

and

$$N^{(k)}(f, z_0) = \underbrace{(N \circ N \circ \cdots \circ N)}_{k \text{ times}}(f, z_0).$$

Newton's method is used to approximate roots of a function.

Newton Iteration

Definition

Let $z \in \mathbb{C}$, then the sequence of Newtons iterates is the sequence $(z_0, z_1, \ldots, z_n)$ defined by $z_{n+1} := N(f, z_n)$. Also, if this sequence satisfies

$$|z_n - \zeta| \leq \left(\frac{1}{2}\right)^{2^n - 1} |z_0 - \zeta|$$

for some true root ζ of f then we call z_o an **approximate root** of f.

Smale, 1981

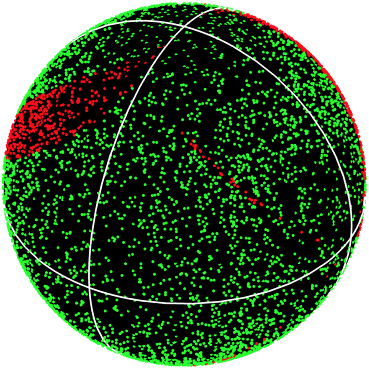
Theorem

If f is a polynomial and $z \in \mathbb{C}$ such that

$$\alpha(f, z) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.157671$$

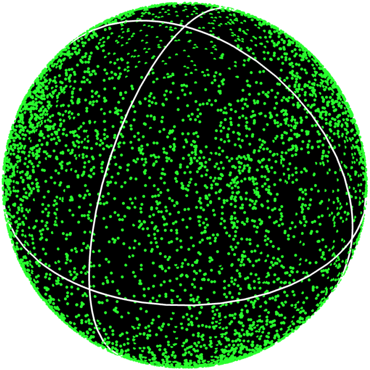
then z is an **approximate solution** to f.

Input: $f(x) = c_0 + c_1x^2 + c_2x^3$



At least one root satisfy α-invariant: Green(Yes) & Red (No)

After 1 Newton iteration



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Conjecture

Conjecture 1

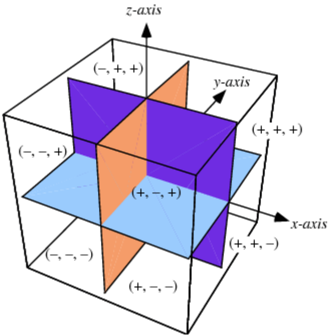
After two iterations of Newton's method at least one root of the lower binomials of $f(x)$ will satisfy the α condition.

Positive Roots of Trinomial and its Lower Binomials

Question 2

How often # of positive roots of lower binomials equal to # of positive roots of f?

Notation



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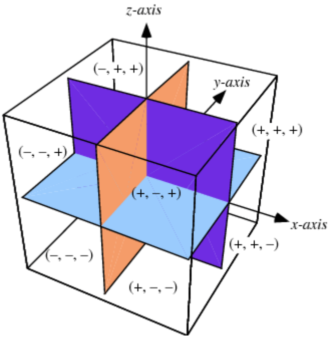

- “good polynomials” vs “bad polynomials”

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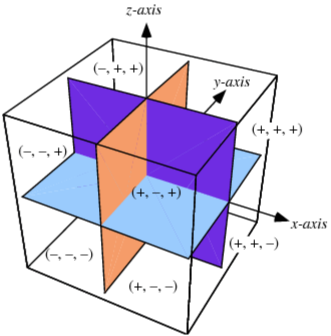

- “good polynomials” vs “bad polynomials”
- Octants are denoted by $\mathcal{O}_{number}$

Descartes' Rule of Sign

Theorem

The number of positive roots of a polynomial is either equal to the number of sign differences between consecutive coefficients, or is less than it by an even number. Multiple roots of the same value are counted separately.

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This is the simplified version of Descartes' rule for positive roots.

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When the number of positive roots of the lower binomials is equal to the number of positive roots of the trinomial?

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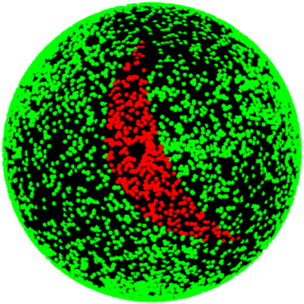

Figure: $a_1 = 10$ and $a_2 = 39$

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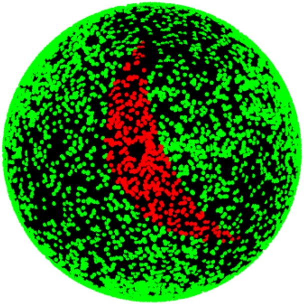

Figure: $a_1 = 10$ and $a_2 = 39$

Successful cases: 94%!!

Probabilities

Can we estimate what are the chances to "hit" a trinomial with as many positive roots as its lower binomials?

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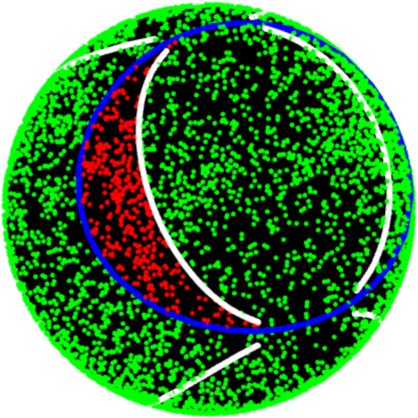
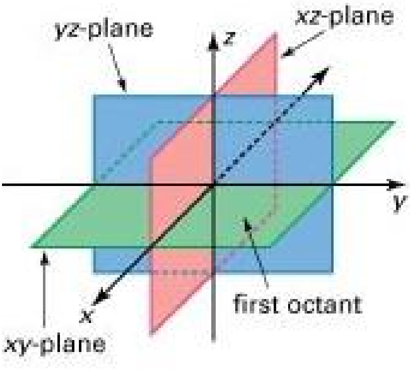


Location, Location, Location

What can we say about the location of the “bad” cases?



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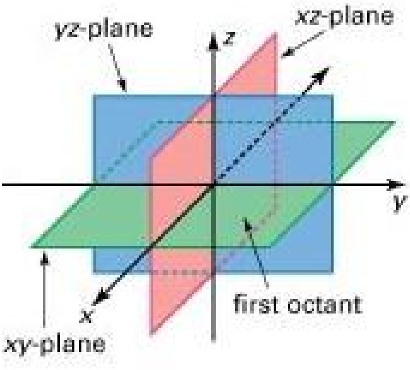

$$\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x^{10} + \frac{\sqrt{3}}{3}x^{39}$$

Clearing octants

x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

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x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- $-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x^{10} + \frac{\sqrt{3}}{3}x^{39}$

Clearing octants

x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- $-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x^{10} + \frac{\sqrt{3}}{3}x^{39}$
- One sign change - Good

Clearing octants

x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- $-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x^{10} + \frac{\sqrt{3}}{3}x^{39}$
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- Two sign changes - Not so good

Clearing octants

x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
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- One sign change - Good
- Two sign changes - Not so good
- The trouble octants are $\mathcal{O}_{IV}$ and $\mathcal{O}_{VIII}$
- The probability is, at least, 75%.

Back to the Future

Conjecture 2

The probability of choosing a randomly generated polynomial with as many positive roots as its lower binomials is, at least, 91.7%.

Upcoming Goals

- Prove **Conjecture 1** and **Conjecture 2**.

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- Use different distributions to generate the coefficients.

Acknowledgments

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